110AH Final Review Problems

Colin Ni

December 10, 2023

Star means highly recommended.

Problem 1*. Let $n \geq 3$. Construct an injection $D_{2n} \hookrightarrow S_n$. Prove or disprove: S_n is the smallest symmetric group into which D_{2n} embeds.

Problem 2*. Let A and B be abelian groups. Denote by $\mathrm{Hom}(A, B)$ the set of group homomorphisms $A \to B$.

(a) Explain how $\mathrm{Hom}(A, B)$ is naturally an abelian group.

(b) Describe $\mathrm{Hom}(\mathbb{Z}, B)$ and $\mathrm{Hom}(C_n, B)$.

(c) In particular, for A and B cyclic, compute $\mathrm{Hom}(A, B)$.

Problem 3*. A theorem of Gauss says that $(\mathbb{Z}/n\mathbb{Z})^\times$, where $n \geq 1$, is cyclic if and only if n is 1, 2, 4, or p^k or $2p^k$ for some odd prime p and $k > 0$. Use this to help fill out the following table of information about $(\mathbb{Z}/n\mathbb{Z})^\times$:

n	cyclic	order	structure	gens	# gens	min size gen set
3						
4						
5						
6						
7	yes	6	C_6	$1, 5$	2	1 (e.g. $\{5\}$)
8	no	4	$C_2 \times C_2$	0	none	2 (e.g. $\{3, 5\}$)
9						
10						
11						
12						
24						
122						
1125						
7938						

Problem 4*. Find the smallest $n \geq 1$ where S_n has an element of order $5n$.

Problem 5. Let p be an odd prime. Show that the only groups of order $2p$ are C_{2p} and D_{2p}.

Problem 6. Is the following 4×4 sliding tile puzzle solvable?:

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

Problem 7*.

(a) Show that every dihedral group has an index 2 subgroup, and generalize this to exhibit an infinite nonabelian group that has an index 2 subgroup.

(b) Denote by S_∞ the group of permutations of $\mathbb{N}$, where $S_n \hookrightarrow S_\infty$ in the natural way. A theorem of Schreier-Ulam says that the only proper nontrivial normal subgroups of S_∞ are $\bigcup_{n\geq 1} S_n$ and $\bigcup_{n\geq 1} A_n$. Use this to show that S_∞ does not have an index 2 subgroup.

(c) (Optional) Show that the only groups whose proper nontrivial subgroups all have index 2 are the simple cyclic groups, C_4, and $C_2 \times C_2$.

Problem 8. Prove, or disprove and find a minimal counterexample:

- If G is a finite group and $d \mid |G|$, then G has an element of order d.
- If G is a finite group and $d \mid |G|$, then G has a subgroup of order d.

You may use that the list of non-abelian groups in increasing order starts with $D_6, D_8, Q_8, D_{10}, D_{12}, A_4, \ldots$.

Problem 9*.

(a) Show that if $S \subset G$ is a normal subset of a group, i.e. $gSg^{-1} \subset S$ for all $g \in G$, then $\langle S \rangle$ is normal.

(b) Show that $A_{3 \cdot 5^2 \cdot 19}$ is generated by the permutations of the form

$$(a_1\ a_2\ a_3)(b_1\ b_2\ b_3\ b_4\ b_5)(c_1\ c_2\ c_3\ c_4\ c_5)(d_1\ d_2\ \cdots\ d_{18}\ d_{19})$$

where the a_i, b_i, c_i, d_i are pairwise distinct.

(c) Show that a nontrivial simple group is generated by its elements of order p if and only if contains an element of order p.

Problem 10. A group G is said to be k-abelian if $(ab)^k = a^k b^k$ for every $a, b \in G$. Show that if a group G is k-, $(k+1)$-, and $(k+2)$-abelian for some $k \in \mathbb{Z}$, then G is abelian.

Problem 11. Let p be an odd prime. The *Legendre symbol* $\left(\frac{-}{p}\right) : (\mathbb{Z}/p\mathbb{Z})^* \to \{\pm 1\}$ is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} +1 & a \text{ is a square in } (\mathbb{Z}/p\mathbb{Z})^* \\ -1 & a \text{ is not a square in } (\mathbb{Z}/p\mathbb{Z})^*. \end{cases}$$

Prove that $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$ for any $a, b \in (\mathbb{Z}/p\mathbb{Z})^*$.

Problem 12*. Let $G \leq \mathbb{C}^*$ the group of p-power roots of unity, where p is a fixed prime. Show that there exists a nontrivial $N \trianglelefteq G$ such that $G \cong G/N$

Problem 13. For which n, m can S_n be embedded into A_m?

Problem 14*. A group G is *finitely generated* if there exists a finite set $S \subset G$ such that $G = \langle S \rangle$. Obviously finite groups are finitely generated, so let us examine infinite groups.

(*a*) Show that $\mathbb{Z}^n$ is finitely generated.

(*b*) Show that $\mathbb{Q}$ is not finitely generated because its finitely generated subgroups are cyclic.

(*c*) Show that $\mathbb{R}$ is not finitely generated but that it has finitely generated subgroups that are not cyclic.

(*d*) Show that the finitely generated group $\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \leq \mathrm{GL}_2(\mathbb{Q})$ has a subgroup that is not finitely generated, namely the one consisting of the matrices in the group with ones on the diagonal.

Remark. A theorem of Higman, Neumann, and Neumann says that every countable group can be embedded into a group generated by two elements.

Problem 15. Given a set of symbols S and a set of relations R which are words in these symbols, the group $\langle S \mid R \rangle$ is the quotient of the free group generated by S by the normal subgroup generated by R. Find a presentation of the groups $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, and $\mathbb{Z} \times \mathbb{Z}$.

Problem 16. Denote by

$$Q_8 = \left\langle -1, i, j, k \;\middle|\; \begin{matrix} (-1)^2 = 1 \\ i^2 = j^2 = k^2 = ijk = -1 \\ -1 \text{ is central} \end{matrix} \right\rangle$$

the *quaternion group*. For $G \in \{Q_8, D_8\}$ do the following:

(*a*) Show that $|G| = 8$, and write down the multiplication table of G.

(*b*) Determine the subgroup lattice of G, and optionally for each subgroup determine its normalizer.

(c) Find all 2-element subsets $S \subset G$ such that $\langle S \rangle = G$.

(d) For each $N \trianglelefteq G$, compute the isomorphism class of G/N.

(e) Determine the conjugacy classes of G.

Problem 17*. (Do Problem 16 first, or look at the answers to it in Solutions.) Let G be a finite group. Prove or disprove:

(a) If all subgroups of G are normal, then G is abelian.

(b) There exists $H, K \lneq G$, one normal, such that $G = HK$ and $H \cap K = 1$.

(c) There exists an injection $G \hookrightarrow S_{|G|-1}$.

(d) If $H \leq G$, then there exists $N \trianglelefteq G$ such that $G/N \cong H$.

(e) If $N \trianglelefteq G$, then there exists $H \leq G$ such that $G/N \cong H$.

(f) If $H, K \trianglelefteq G$ and $G/H \cong G/K$, then $H \cong K$.

(g) If $H, K \trianglelefteq G$ and $H \cong K$, then $G/H \cong G/K$.

Remark. Cayley's theorem exhibits an injection $G \hookrightarrow S_{|G|}$ for any finite group G, so part (c) is asking whether this $|G|$ is sharp.

Problem 18*. Show that a transitive group action is the same thing as left-multiplication on a coset space. More precisely, show that if G acts transitively on a set X, then $X \cong G/G_x$ as G-sets for any $x \in X$.

Problem 19*. Show that a finite group is not the union of the conjugates of one of its proper subgroups.

Problem 20*. Let G be a finite group, and let $d \in \mathbb{N}$. Prove and generalize, or disprove:

(a) If $d \mid |G|$, then G acts transitively on a set with d elements.

(b) If $|G| = 144$, then G acts transitively on a set with 9 elements.

Problem 21. A group G is *solvable* if there exist subgroups

$$1 = N_1 \trianglelefteq N_2 \trianglelefteq \cdots \trianglelefteq N_{r-1} \trianglelefteq N_r = G$$

such that N_{i+1}/N_i is abelian for $i = 1, \ldots, r-1$. Prove the following using the isomorphism theorems:

(a) A subgroup of a solvable group is solvable.

(b) The homomorphic image of a solvable group is solvable.

(c) Show that if $N \trianglelefteq G$ and G/N are solvable, then G is solvable.

Problem 22. Show that any p-group or any group G with order pq, p^2q, p^2q^2, or pqr where p, q, r are primes is solvable.

Problem 23. Recall that

$$S_n \cong \left\langle x_1, \ldots, x_{n-1} \;\middle|\; \begin{array}{c} x_i^2 \text{ for } i = 1, \ldots, n-1 \\ (x_i x_{i+1})^3 \text{ for } i = 1, \ldots, n-2 \\ (x_i x_j)^2 \text{ for } i < j \text{ and } |j-i| > 1 \end{array} \right\rangle$$

via the isomorphism $\tau_i = (i\ i+1) \leftarrow\!\shortmid x_i$.

(a) Two triple transpositions in S_6 share 0, 1, 2, or 3 transpositions. In each case, what is the cycle type of their product?

(b) Find an automorphism $S_6 \to S_6$ that takes transpositions to triple transpositions, and hence is not an inner automorphism.

Problem 24*. Let G be a group. The *commutator* of $x, y \in G$ is defined to be $[x, y] = xyx^{-1}y^{-1}$, and the commutator subgroup $G' \leq G$ is the subgroup generated by all commutators.

(a) Show that G is a abelian if and only if $G' = 1$.

(b) Show that G' is the smallest normal subgroup with abelian quotient, i.e. if $N \trianglelefteq G$ and G/N is abelian, then $G' \leq N$.

(c) Show that any subgroup containing G' is normal.

Problem 25. Show that a proper subgroup of a p-group is properly contained in its normalizer.

Problem 26*. Compute the order of the normalizer $N_{S_p}(C)$ where $C \leq S_p$ is a cyclic subgroup of order p.

Problem 27. Let G be a finite group and X a finite G-set. Prove Burnside's lemma:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|.$$

Deduce that a finite group acting transitively on a non-singleton set has a fixed-point-free element.

Problem 28*.

(a) (Optional) Prove the following extension of Bézout's identity: For $a, b \in \mathbb{N}$ coprime and $c \geq (a-1)(b-1)$, there exists $x, y \geq 0$ such that $ax + by = c$.

(b) Let G be a finite group of order 35. Determine the set of the sizes of the finite G-sets with no fixed points. Optionally, generalize.

Problem 29*. Let H be a nontrivial p-group for some prime p.

(a) Show that the center of H is nontrivial, using that the size of a conjugacy class in a finite group divides the order of the group.

(b) Write $|H| = p^n$ for some $n \geq 1$. Show that H has a subgroup of order p^k for every $0 \leq k \leq n$.

(c) Suppose H injects into a finite group G with coprime order. Prove and generalize, or disprove and fix: H contains all elements in G that have order p.

Problem 30. Suppose G is a finite simple group that has a proper subgroup of index n. Recall that $|G| \mid n!$. Show that in fact $|G| \mid \frac{1}{2}n!$.

Problem 31. (Optional) The homophonic group H is the group generated by the 26 letters of the English alphabet modulo homophones, i.e. two English words with the same pronunciation are equal in H. Show that H is trivial.

Problem 32. Let G be a group, and let $S, T \leq G$ be subgroups.

(a) Show that $ST = TS$ if and only if $ST \leq G$ if and only if $TS \leq G$.

(b) Show that if S or T is normal, then equivalent statements in part (a) hold.

Problem 33*. Let G be a group with $N \trianglelefteq G$ and $H \leq G$. Show that the following definitions for G being the inner semidirect product of N and H are equivalent:

(i) $G = NH$ and $N \cap H = 1$

(i)$'$ $G = HN$ and $H \cap N = 1$

(ii) for every $g \in G$, there exists unique $n \in N$ and $h \in H$ such that $g = nh$

(ii)$'$ for every $g \in G$, there exists unique $h \in H$ and $n \in N$ such that $g = hn$

(iii) $H \hookrightarrow G \to G/N$ is an isomorphism

Problem 34*. Show that D_{2n}, where $n \geq 3$, is a nontrivial semidirect product but that neither C_4 nor Q_8 is.

Problem 35. Let p be a prime, set $X = \{1, \ldots, p\}$, and let $G \leq S_p$ be transitive.

(a) Show that G acts on X transitively if and only if G has a Sylow p-subgroup.

(*b*) Define n_G and r_G for a Sylow p-subgroup $P \leq G$ as follows:

$$\begin{tikzcd}
& S_p \arrow[dl, dash] \arrow[dr, dash] & \\
G \arrow[dr, dash, "n_G"'] & & N_{S_p}(P) \arrow[dl, dash] \\
& N_G(P) \arrow[d, dash, "r_G"'] & \\
& P \arrow[d, dash, "p"'] & \\
& 1 &
\end{tikzcd}$$

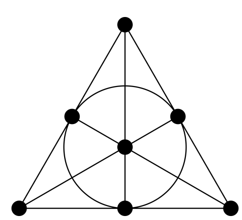
Show that n_G and r_G are independent of the Sylow p-subgroup $P \leq G$. Note that $|G| = n_G r_G p$ and that $r_G \mid (p-1)$ by Problem 26.

(*c*) Show that if $r_G = 1$, then $G \cong C_p$.

(*d*) Suppose $|G| = nrp$ where $r < p$ is also prime, $n > 1$, and $n \equiv 1 \mod p$. Show that $r = r_G$ and $n = n_G$. Moreover, show that any nontrivial $N \trianglelefteq G$ is transitive and that $n_N = n$ and $r_N = r$. Deduce that G is simple.

Problem 36. A *Steiner system* $S(\ell, m, n)$ for positive integers $\ell < m < n$ is a collection of distinct size-m subsets of $\{1, \dots, n\}$ called *blocks* such that every size-ℓ subset of $\{1, \dots, n\}$ is contained in exactly one block. The automorphism group $\mathrm{Aut}(S(\ell, m, n))$ is the subgroup of S_n taking blocks to blocks.

(*a*) Explain how the following picture depicts a $S(2, 3, 7)$:



(*b*) Suppose there exists a $S(\ell, m, n)$ for some $\ell \geq 2$. Show that there exists a $S(\ell-1, m-1, n-1)$ such that its automorphism group is a stabilizer subgroup of the action of $S(\ell, m, n)$ on $\{1, \dots, n\}$. Moreover, show that if $\mathrm{Aut}(S(\ell, m, n))$ is k-transitive, then $\mathrm{Aut}(S(\ell-1, m-1, n-1))$ is $(k-1)$-transitive.

(*c*) There exists a unique $S(5, 6, 12)$ and a unique $S(5, 8, 24)$. Denote by M_{24} and M_{12} their automorphism groups which are both 5-transitive and which are called *Mathieu groups*. Spam part (b) to fill out or make sense of the first three columns of the following table:

group	order	transitivity	simple	sporadic
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	5		yes
M_{23}				yes
M_{22}				yes
M_{21}			yes	no
M_{20}			no	no
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	5		yes
M_{11}				yes
M_{10}			no	no
M_9			no	no
M_8			no	no

(d) Show that M_{24}, M_{23}, M_{22}, M_{12}, and M_{11} are simple, using that M_{21} is simple (but not sporadic), Problem 35, and the following simplicity criterion, which is Theorem 9.25 in Rotman's *Introduction to the Theory of Groups*. Let X be a faithful k-transitive G-set for some $k \geq 2$, and assume G has a simple stabilizer subgroup. Then the following are true:

- If $k \geq 4$, then G is simple.
- If $k \geq 3$ and $|X|$ is not a power of 2, then $G \cong S_3$ or G is simple.
- If $k \geq 2$ and $|X|$ is not a prime power, then G is simple.